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Theory of the Control of Structures by Low-Authority Controllers

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Structural vibrations can be controlled by low-gain controllers which moderately modify the structure's characteristics. Typically, 10-20% damping may be obtained. The study of such controllers is made using perturbation methods. A new formula is established for the eigenvectors perturbation which helps define the validity range of the well-known Jacobi's root perturbation formula. Analytical formulas are then derived relating the root shifts and mode shape changes to the controller's gains, and can be used either for prediction of performance or for design of the controller. Numerical examples of such a design are given for simple beam structures with linear and rotational sensors and actuators.

Introduction

In recent years, spacecraft structural flexibility has been of growing concern to space vehicle designers, not only because of its interaction with the main attitude control system, as has been the case in the past, but more directly because of the active role played by these structures in mission requirements. This is typical of large antennas and optical systems, solar powerplants, etc., all involving the accurate control of large structures in space. This multivariable control problem has created a whole new field of investigation, not only for modern control theory, but also for structural analysis concepts and applications.

Because of the numerical complexity of any finite-element model, it is usual to first reduce numerically the data into a limited number of mode shapes and frequencies. Although more tractable, the dynamic models based on this representation still involve a large number of differential equations in most cases of practical interest. Modern control theory has developed sophisticated ways to handle these high-order systems, but it has to resort to numerical methods which cannot usually keep track of the functional dependency between the performance and the model parameters other than via lengthy parametric studies.

A problem of interest, and one which is addressed in this paper, concerns methods by which the natural (and generally low) damping of a structure may be augmented by adding to it energy dissipation devices (so-called dampers). While it is relatively easy to model such devices and analyze the characteristics of the total system, an a priori prediction of their effect is not feasible with the methods mentioned above. This fact was clearly mentioned, for instance, in a recent state-of-the-art survey by Draper Laboratory.[1]

The fundamental reason for this unpredictability comes from the fact that no analytical expression exists for the roots of high-order polynomials, and even less so for the eigenvectors of large matrices. Thus any modification of the original system, as occurs for instance when damping devices are implemented in a structure, will lead to a new set of eigenvalues/eigenvectors that need to be determined all over again by numerical processes. As stated earlier, this a posteriori evaluation is not very convenient for design purposes, and the original decision on locating and sizing the dampers is pretty much based on guesswork and engineering judgment.

The novel idea presented in this paper is based on the observation that if a structure is controlled by distributed systems of sensors and actuators with limited authority, i.e., if the controller is allowed to modify only moderately the natural modes and frequencies of the structure, then it should be possible to apply root perturbation techniques to predict analytically the behavior of the total system. For instance, starting with an undamped structure and placing dampers in it in such a way that, say, 10% system damping is obtained in the principal modes, implies that the relative change $|d\lambda_n/\lambda_n|$ of the corresponding complex root $\lambda_n = i\omega_n$ is only 10%, which is small enough to justify first-order expansions. Of course, only approximating analytical expressions can be derived in this way, but they may suffice in practical cases and will provide at least a very strong rationale for the first step in the design iteration loop.

The idea of root perturbation is not new, however. It was first introduced by Jacobi in 1846[2] and has since been applied extensively, but mostly to evaluate root sensitivity to system parameter changes. (See, for instance, Refs. 3 and 4.) The method proposed here could be interpreted also as a root sensitivity analysis, where the "gains" of the control system are the parameters changing about a nominal zero value.

Jacobi's Formula and Further Generalizations

This section will outline the basic formulas of the eigensystem perturbation theory. The root perturbation formula was first derived by Jacobi for infinitesimal perturbations which neglect the induced eigenvector perturbation. Formulas for perturbing the eigenvectors are developed here, leading to a more general form of Jacobi's formula.

Consider first a square matrix A and one of its eigenvalues, say λ_n, and the corresponding left and right eigenvectors L_n and R_n defined by

$$AR_n = \lambda_n R_n \tag{1}$$

$$L_n^T A = \lambda_n L_n^T \tag{2}$$

Assuming now a change δA in the coefficients of the matrix A so that A becomes $A' = A + \delta A$, the quantities λ_n and R_n change correspondingly by $\delta\lambda_n$ and δR_n such that one now obtains

$$(A+\delta A)(R_n+\delta R_n) = (\lambda_n+\delta\lambda_n)(R_n+\delta R_n)$$

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which, after substituting from Eq. (1), reduces to

$$A\delta R_n + \delta A(R_n + \delta R_n) = \lambda_n \delta R_n + \delta\lambda_n (R_n + \delta R_n) \tag{3}$$

Multiplying the matrix equation (3) on the left by L_n^T and using Eq. (2), one obtains the exact formula

$$\delta\lambda_n = \frac{L_n^T \delta A (R_n + \delta R_n)}{L_n^T (R_n + \delta R_n)} \tag{4}$$

Jacobi's formula can now be obtained from Eq. (4) by assuming δR_n to be sufficiently small to be ignored and writing "$\delta \to$ d":

$$\mathrm{d}\lambda_n = L_n^T \mathrm{d}A R_n / L_n^T R_n \qquad \text{(Jacobi)} \tag{5}$$

The perturbation in the eigenvector itself is less obvious to obtain, due to the fact that an eigenvector is only defined within a multiplicative constant. Thus some normalization must be introduced so that the eigendirection δR_n can be uniquely defined. To accomplish this, another perturbation formula is obtained by multiplying Eq. (3) on the left by $R_n^T G_n$, where G_n is a normalization weighting matrix such that

$$R_n^T G_n R_n = 1$$

Thus, for sufficiently small δR_n, denoted by $\mathrm{d}R_n$, we have

$$R_n^T G_n \mathrm{d}R_n = 0 \tag{6}$$

[Note: Because R_n is a *complex* vector, it could be normalized by dividing it by its length $\|R_n\|$ and denoted by $\hat{R}_n$, where $\|$ denotes the usual Hermitian norm. If (,) then denotes the corresponding Hermitian inner-product, it is generally true, in contradistinction to the real case, that $(\hat{R}_n, \mathrm{d}\hat{R}_n) \neq 0$. This motivates the use of the weighting matrix G_n.]

After expanding Eq. (3), multiplying on the left by $R_n^T G_n$, and using Eq. (6), one obtains

$$\mathrm{d}\lambda_n \cong R_n^T G_n \mathrm{d}A R_n + R_n^T G_n A \mathrm{d}R_n$$

This expression is then used to eliminate $\mathrm{d}\lambda_n$ in Eq. (3), leading to

$$\mathrm{d}R_n = (A - R_n R_n^T G_n A - \lambda_n I)^{-1} (R_n R_n^T G_n - I) \mathrm{d}A R_n \tag{7}$$

As shown in Appendix A, the matrix inversion appearing here is always possible when λ_n has multiplicity 1 and this expression of $\mathrm{d}R_n$ automatically satisfies Eqs. (6) and (3).

This is an analytical result much simpler than those previously proposed.[5,6] A proper choice of the matrix G_n may also simplify the computation. For a purely numerical solution, however, a simpler method could be used.[7] The second-order perturbation of the eigenvalues may then be obtained by using Eq. (7) in Eq. (4), rewritten with $\delta \to$ d.

First-Order Structural Equations and Modal State Vectors

To apply the perturbation formulas to the control of a structure, it is necessary to transform first the classical undamped structural equations

$$M\ddot{u} + Ku = F \tag{8}$$

into the classical modal equations

$$\ddot{q} + \left[\diagdown \omega_n^2 \diagdown\right] q = \Phi^T F \tag{9}$$

where[8]:

$$\begin{cases} u = \Phi q \\ \Phi^T M \Phi = I \\ \Phi^T K \Phi = \left[\diagdown \omega_n^2 \diagdown\right] \end{cases}$$

Equation (9) is then rewritten in first-order form by defining the N-dimensional state-vector ($N = 2N_m$)

$$X \equiv [\dot{q} \;\; q]^T$$

so that one obtains

$$\dot{X} = AX + \begin{bmatrix} \Phi^T \\ \hline 0 \end{bmatrix} F$$

where

$$A \equiv \left[\begin{array}{c|c} 0 & -\diagdown \omega_n^2 \diagdown \\ \hline I & 0 \end{array}\right] \tag{10}$$

Complex eigenvalues and eigenvectors are now associated with the matrix A. [Note: the eigenvectors of A (in the usual sense) are called the *right eigenvectors of A* because the eigenproblem is written as $AR_n = \lambda_n R_n$. The eigenvectors of A^T (in the usual sense) are called the *left eigenvectors of A* because the eigenproblem $A^T L_n = \lambda_n L_n$ can also be written $L_n^T A = \lambda_n L_n^T$. These two types of eigenvectors eliminate the simultaneous use of A and A^T.]

The eigenvalues of A are $\lambda_n = \pm i\omega_n$ ($i^2 = -1$, and $n = 1,2,\ldots,N_m$) and the *modal state eigenvectors* are defined to be the right and left eigenvectors of A corresponding to $\lambda_n = +i\omega_n$, i.e.,

$$\begin{aligned} R_n &\equiv [\; 0\; 0 \ldots 0 \; i\omega_n \; 0 \ldots 0 \mid 0\; 0 \ldots 0 \; 1 \; 0 \ldots 0 \;]^T \\ L_n &\equiv [\; 0\; 0 \ldots 0 \; 1 \; 0 \ldots 0 \mid 0\; 0 \ldots 0 \; i\omega_n 0 \ldots 0 \;]^T \end{aligned} \tag{11}$$

where the only nonzero elements occur in position n and $n + N_m$.

State-Space Equations for Damper-Augmented Structures

Damping may be introduced in the structure by measuring (components of) velocities $\dot{u}_r$ at some sensor location r ($r = 1,2,\ldots,N_r$) and producing forces F_a at actuator locations a ($a = 1,2,\ldots,N_a$) such that, in general

$$F_a = -\sum_r D_{ar} \dot{u}_r \tag{12}$$

where the D_{ar}'s are positive (rate-feedback) gains, collectively denoted by the matrix D which is augmented by zeros to be dimensionally compatible with the modal matrix Φ. The structural Eqs. (10) then become

$$\dot{X} = A' X \tag{13}$$

where

$$A' = A + \mathrm{d}A \equiv \left[\begin{array}{c|c} -\Phi^T D\Phi & -\omega_n^2 \\ \hline I & 0 \end{array}\right]$$

The new roots of the linear system [Eq. (13)] may then be obtained by considering the matrix

$$\mathrm{d}A \equiv \left[\begin{array}{c|c} -\Phi^T D\Phi & 0 \\ \hline 0 & 0 \end{array}\right] \tag{14}$$

as a perturbation of the original matrix A and applying Jacobi's formulas.

Modal Damping (Rate-Feedback) Prediction Formulas

To the original (undamped) mode of frequency $\omega_n/2\pi$ corresponds the eigenvalue $\lambda_n = i\omega_n$. If the state-space matrix A in Eq. (10) is now perturbed by $\mathrm{d}A$ by the introduction of "moderate" damping (rate-feedback) according to Eq. (14), the original mode will now have the eigenvalue $\lambda_n + \mathrm{d}\lambda_n$, where, by Jacobi's formula (5), $\mathrm{d}\lambda_n$ is given by

$$\mathrm{d}\lambda_n \cong L_n^T \mathrm{d}AR_n / L_n^T R_n$$

From Eq. (11) we have

$$L_n^T R_n = 2i\omega_n$$

For brevity, define the matrix $B = [B_{kn}]$ and the vector $Z = [Z_1, Z_2, \ldots, Z_n]^T$ by

$$B \equiv \Phi^T D\Phi \qquad Z \equiv \mathrm{d}AR_n \tag{15}$$

Then from Eqs. (11) and (14) we obtain:

$$Z_k = \begin{cases} -i\omega_n B_{kn} & \text{for} \quad k = 1,2,\ldots,N_m \\ 0 & \text{for} \quad k > N_m \end{cases}$$

and thus

$$L_n^T Z = -iB_{nn}\omega_n$$

Consequently, Jacobi's formula leads to

$$\mathrm{d}\lambda_n \cong -\frac{1}{2}B_{nn} = -\frac{1}{2}\sum_{a,r} D_{ar}\phi_{an}\phi_{rn}$$

where ϕ_{an} and ϕ_{rn} are the elements of the modal matrix Φ corresponding to the nth mode, at actuator nodal stations a and sensor nodal stations r. Since the ϕ's and D's (positive rate-feedback gains) are real, it follows that the new root $\lambda_n + \mathrm{d}\lambda_n$ has now a real part, and with the usual interpretation of damping coefficient we obtain the fundamental modal damping prediction formula:

$$2\zeta_n\omega_n \cong \sum_{a,r} D_{ar}\phi_{an}\phi_{rn} \tag{16}$$

The above is thus a general formula for predicting modal damping in a rate-feedback system of distributed sensors ($r = 1,2,\ldots,N_r$) and actuators ($a = 1,2,\ldots,N_a$).† It confirms the intuitive idea that sensors and actuators should be located preferably where large modal deflections occur, i.e., where mode shapes have large magnitude. Obviously, if ϕ_{rn} is zero, mode #n is not observable and if ϕ_{an} is zero, it is not controllable. Moreover, Eq. (16) gives quantitative estimates of modal damping and stability. A very significant consequence of Eq. (16) is that when $a = r$, i.e., when sensors and actuators are colocated, the structure is always stable since, for positive D's, the sum on the right-hand side is always positive for any n.

†In a three-degree-of-freedom nodal structure, the indices a and r appearing in Eqs. (12) and (16) refer in fact to a particular coordinte at a partiular node. For instance F_8 may denote the y component of a force actuator at some station, while F_9 is the z component at the same station.

General Output Feedback Case

It may be desirable in some cases to introduce position feedback in the controller. For instance, the "rigid body modes" are usually taken care of by the attitude control system. The general feedback law is now written as

$$F_a = -\Sigma_r(D_{ar}\dot{u}_r + C_{ar}u_r) \tag{17}$$

This time, the C term will produce a first-order shift $\mathrm{d}\omega_n$ in the modal frequencies. Formulas may be derived in a fashion similar to the previous section leading to the following results. Defining

$$B_{nm} \equiv \sum_{a,r} D_{ar}\phi_{an}\phi_{rm} \qquad Q_{nm} \equiv \sum_{a,r} C_{ar}\phi_{an}\phi_{rm} \tag{18}$$

the design/prediction formulas are then, for $\omega_n \neq 0$,

$$\zeta_n \simeq B_{nn}/2\omega_n \qquad \mathrm{d}\omega_n \simeq Q_{nn}/2\omega_n \tag{19}$$

For rigid-body modes, where ω_n is initially zero, and for moderate damping, one obtains instead

$$\zeta_n \simeq \tfrac{1}{2}B_{nn}/\sqrt{Q_{nn}} \qquad \delta\omega_n \simeq \sqrt{Q_{nn}} \tag{20}$$

Complex Mode Shapes and Second-Order Formulas

While in the undamped structure the mode shapes are real, i.e., all the points of the structure move in synchrony, this ceases to be true in general when active control is introduced. Instead, the closed-loop mode shapes are characterized by the fact that there are phase shifts associated with each point of the structure. Thus the new mode shape may be decomposed in an in-phase part (real part) and a 90 deg out-of-phase part (imaginary part)

$$\psi_n = \psi_n^R + i\psi_n^I \tag{21}$$

Since $u = \Phi q$, ψ_n is the value of u when q is equal to the nth eigensolution of the closed-loop Eq. (13), which is simply $R_n^o + \delta R_n^o$, where R_n^o is a partition of R_n containing the last N_m components of R_n:

$$R_n = [R_n' \mid R_n^o]^T$$

Thus

$$\psi_n = \Phi(R_n^o + \delta R_n^o)$$

The first product in the above equation is simply equal to the nth mode shape ϕ_n because of the particular structure of R_n^o [see Eq. (11)]. Thus finally,

$$\psi_n = \phi_n + \Phi\delta R_n^o \tag{22}$$

An approximate expression for the vector δR_n^o may be calculated explicitly using Eq. (7) in which the matrix G_n is chosen of the form

$$G_n = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \frac{1}{2\omega_n^2} & & \\ & & & \ddots & \\ & & & \frac{1}{2} & \\ & & & & \ddots \;\; 0 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{row } n \\ \\ \leftarrow \text{row } 2n \\ \\ \end{matrix}$$

Defining

$$\epsilon_{nm} = Q_{mn}/(\omega_n^2 - \omega_m^2) \qquad \epsilon'_{nm} = \omega_n B_{mn}/(\omega_n^2 - \omega_m^2) \qquad (n \neq m)$$

$$\epsilon_{nn} = Q_{nn}/4\omega_n^2 \qquad \epsilon'_{nn} = -B_{nn}/4\omega_n \qquad (n = m) \tag{23}$$

the perturbations of the eigenvector $R_n = [R'_n \mid R^o_n]^T$ may be written componentwise as

$$\delta R^o_{mn} = \epsilon_{nm} + i\epsilon'_{nm}$$

$$\delta R'_{mn} = \begin{cases} i\omega_m \delta R^o_m & \text{if} \quad m \neq n \\ -\omega_n \delta R^o_n & \text{if} \quad m = n \end{cases} \tag{24}$$

and the complex mode shapes are then

$$\psi_n = \phi_n + \sum_m \epsilon_{nm}\phi_m + i\sum_m \epsilon'_{nm}\phi_m \tag{25}$$

This shows that damping will in general result in complex modes. However, if B and Q happen to be both diagonal [a necessary and sufficient condition for this is that they be linear combinations of the matrices M and K of Eq. (8), a condition usually called "proportional damping"], then

$$\psi_n = (1 + \epsilon_{nm} + i\epsilon'_{nm})\phi_n$$

Since this is just a scalar multiplication, the mode shapes are unchanged in that case and all the points of the structure stay in phase.

Finally, the second-order perturbation for the roots can be obtained by using in Eq. (4) the δR's given in Eq. (24), leading to the formulas

$$2\zeta_n\omega_n = \frac{B_{nn}}{2} - \frac{B_{nn}Q_{nn}}{4\omega_n^2} + \frac{1}{2}\sum_{m\neq n}\frac{(B_{nm}Q_{mn} + Q_{nm}B_{mn})}{\omega_n^2 - \omega_m^2}$$

$$\delta\omega_n = \frac{Q_{nn}}{2\omega_n} + \frac{Q_{nn}^2 - \omega_n B_{nn}^2}{8\omega_n^2} + \frac{1}{2}\sum_{m\neq n}\frac{(Q_{nm}Q_{mn} - \omega_n B_{nm}B_{mn})}{2\omega_n(\omega_n^2 - \omega_m^2)} \tag{26}$$

Extension to Generalized Modes and Force Distributions

So far the theory has been established for sensors measuring displacement (or velocities) and actuators applying local forces. The results may be extended to cases where sensors measure quantities such as local rotations, bending moments, etc., and where actuators produce a distribution of loads (such as doublets provided by torquers, moment actuators, etc.).

The sensor output σ may be defined by a linear transformation of the type

$$\sigma_r = \Sigma_j H_{rj} u_j$$

For instance, in a one-dimensional structure, the local rotation may be expressed as

$$\sigma_r = (u_{r+1} - u_{r-1})/\Delta x$$

where Δx is the distance between stations $r+1$ and $r-1$. Since $u = \Phi q$, the vector of sensor outputs may be written as

$$\sigma = \Phi^R q$$

where

$$\Phi^R \equiv H\Phi$$

is a matrix of generalized mode shapes. (In a three-dimensional structure, "local rotation" mode shapes are obtained by "stacking" the three-dimensional curls $\vec{\nabla} \times \vec{\phi}_n$ of the deformation mode shapes.)

In the same way, an actuator controlled by a single input $f_a(t)$ may produce a distribution of loads:

$$f_{aj} = P_{aj} f_a(t)$$

where $j = 1,\ldots,N_m$ is a station index.

Thus the control equation (18) becomes

$$f_{aj} = -P_{aj}\Sigma_{r,k}(D_{ar}H_{rk}\dot{u}_k + C_{ar}H_{rk}u_k)$$

and the matrices B and Q are now defined by

$$B = \Phi^{A^T} D\Phi^R \qquad Q = \Phi^{A^T} C\Phi^R$$

where the generalized mode shape matrices Φ^A and Φ^R are

$$\Phi^A = P\Phi \qquad \Phi^R = H\Phi$$

Thus, Eq. (19) may be written in a similar way:

$$B_{nm} = \Sigma_{a,r} D_{ar}\phi^A_{an}\phi^R_{rm} \qquad Q_{nm} = \Sigma_{a,r} C_{ar}\phi^A_{an}\phi^R_{rm} \tag{27}$$

and Eqs. (20-26) remain unchanged.

Examples of Application to Active Structural Damping

Design and prediction of active structural damping will be illustrated below on simple cases. The structure considered here is a 100-m-long graphite-epoxy tube which will be either cantilevered or free-floating in space. Its characteristics and mode shapes are given in Appendix B.

Case A

Assuming the beam is cantilevered, design a damper using a force actuator at the free end such that the damping in the first mode is 10%. Since ϕ_1 has maximum value at the free end, the actuator will be located there with its sensor.

The first modal frequency is 0.0605 Hz, thus Eq. (19) gives

$$B_{11} = 2 \times 2\pi \times 0.0605 \times 0.10 = 0.07605$$

Solving Eq. (19) for D_{11}, with $\phi_{11} = 0.12827$, the feedback gain is determined to be 4.622. Now Eq. (19) may be used directly to predict the damping in the other modes. Comparison with the actual values (obtained by a full eigensystem analysis) is given in Table 1.

Case B

In the following example, the previous beam is free in space and is controlled by a torquer ($A1$) situated close to one end (Fig. 1). Two sensors S_1 and S_2 are used, one where the torquer is located, the other one in the middle of the beam.

Table 1 Comparison with actual values

Original frequency	Predicted frequency	Actual frequency	Predicted damping	Actual damping
6.0519 − 02	6.0555 − 02	6.0555 − 02	0.100005	0.100063
3.7927 − 01	3.7911 − 01	3.7911 − 01	0.015422	0.015421
1.0620 + 00	1.0618 + 00	1.0618 + 00	0.005338	0.005338
2.0810 + 00	2.0809 + 00	2.0809 + 00	0.002637	0.002637
3.4401 + 00	3.4400 + 00	3.4400 + 00	0.001543	0.001543
5.1388 + 00	5.1388 + 00	5.1388 + 00	0.000999	0.000999

Table 2 Low-authority controller for 100-m beam design, prediction and evaluation results

	Mode no.	Initial values	Target values	Predicted values	Actual values	
Frequencies	1	0.000		0.072	0.063	(rigid body)
	2	0.385	0.420	0.432	0.432	(bending)
	3	1.061	1.090	1.119	1.124	
	4	2.081		2.271	2.423	
	5	3.440		3.364	3.022	
	6	5.139		4.619	4.776	
	7	7.177		7.122	7.142	
Damping ratios	1	0.000		0.214	0.212	(rigid body)
	2	0.000	0.200	0.186	0.175	(bending)
	3	0.000	0.100	0.089	0.084	
	4	0.000		0.177	0.148	
	5	0.000		0.331	0.450	
	6	0.000		0.117	0.060	
	7	0.000		0.008	0.006	

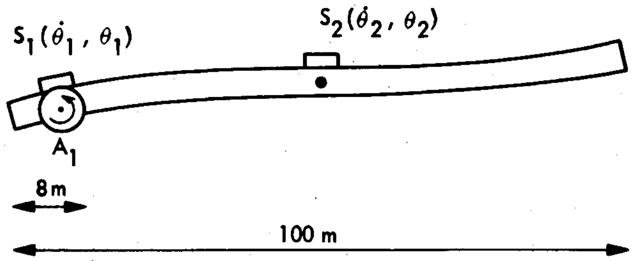

Fig. 1 Beam control system.

These sensors are assumed to measure local angular position and rate θ and $\dot\theta$.

The control law then has the form

$$T_1 = -D_{11}\dot\theta_1 - D_{12}\dot\theta_2 - C_{11}\theta_1 - C_{12}\theta_2$$

and it is possible to specify the frequencies and damping ratios of two arbitrary modes. Since torquers and angular sensors are used, "local rotation modes" must be computed and the extended formulas [Eqs. (27)] may then provide values for D's and C's. For instance, 20 and 10% damping were desired for modes 2 and 3, respectively. These modes are respectively the first symmetric and first antisymmetric mode of the beam, mode 1 being a rigid body mode defining the attitude of the beam. Also, it was desired to raise the frequency of mode 2 to 0.43 Hz and mode 3 to 1.09 Hz. The gains were calculated to be

$$D_{11} = 29.7\ 10^3\text{N}\cdot\text{m}\cdot\text{s/rad} \qquad C_{11} = 31.2\ 10^3\text{N}\cdot\text{m/rad}$$

$$D_{12} = 21.9\ 10^3\text{N}\cdot\text{m}\cdot\text{s/rad} \qquad C_{12} = 88.1\ 10^3\text{N}\cdot\text{m/rad}$$

and the overall results displayed in Table 2 show how the attitude and the first six bending modes of the beam are affected by this control system.

It is interesting to remark that with a force actuator (case A), the *time decays* for all modes are similar, while with the torque actuator the *damping ratios* are comparable. This fact is due to the different behavior, as a function of modal frequency, of the "displacement" and "rotation" mode shapes, and to Eqs. (27). Figure 2 shows time histories of the modal amplitude of the rigid body mode (theta) and the first three bending modes (q_1, q_2, and q_3). The initial conditions correspond to about 1 cm deflection at each end and no initial velocity. Since the rigid body mode was of lower frequency, the time scale is larger for its plot. Finally, an example of the complex mode shapes introduced by the control system is shown in Fig. 3 where the real and imaginary parts of the beam deflection have been plotted for bending modes 2 and 6. In the real part, one may recognize the usual bending modes for a free-free beam. The imaginary part is quite different, however, and shows how the actuator introduces the largest phase shift in its own vicinity (on the left) and this effect dies out at remote points of the structure.

Perturbation Method Prediction Accuracy

Although perturbation methods will give asymptotically correct answers when the perturbation goes to zero, in any practical case where the perturbation is finite, an error will be committed. Thus the usefulness of such methods is greatly improved if the error can be estimated.

By examining Eq. (16), it appears as if only the diagonal elements of $\phi^T D \phi$ have been retained and, therefore, that this approximation will be good only if the off-diagonal elements are small. The situation is not so simple, however, and the correct answer must be obtained from the shift formula [Eq. (4)] which shows that the error in root shift prediction committed when using Jacobi's formula [Eq. (5)] is a function of the eigenvector shift δR_n. Although δR_n cannot be computed exactly, its approximate value may be obtained from Eqs. (23) and (24).

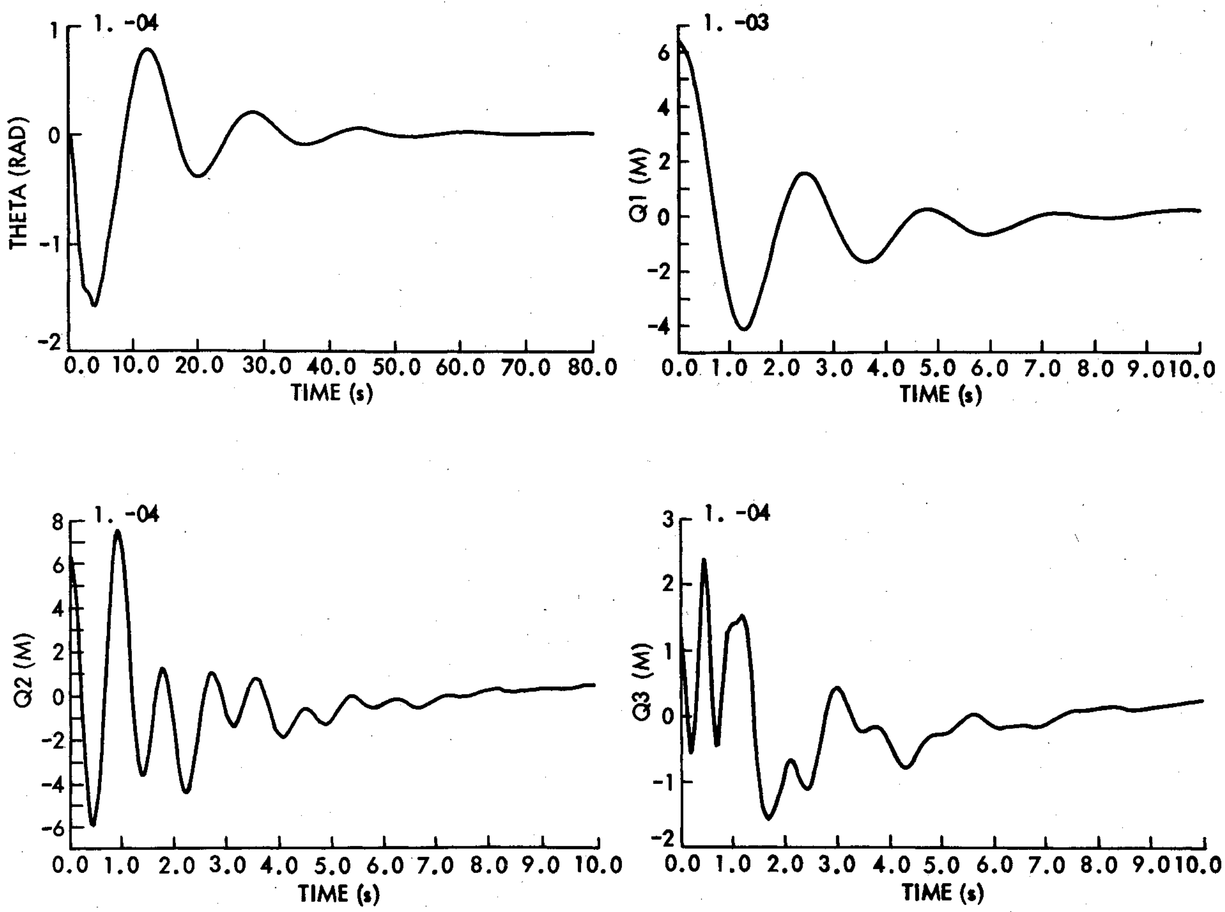

Fig. 2 Time histories of attitude angle and first three bending modes of the damped 100-m beam.

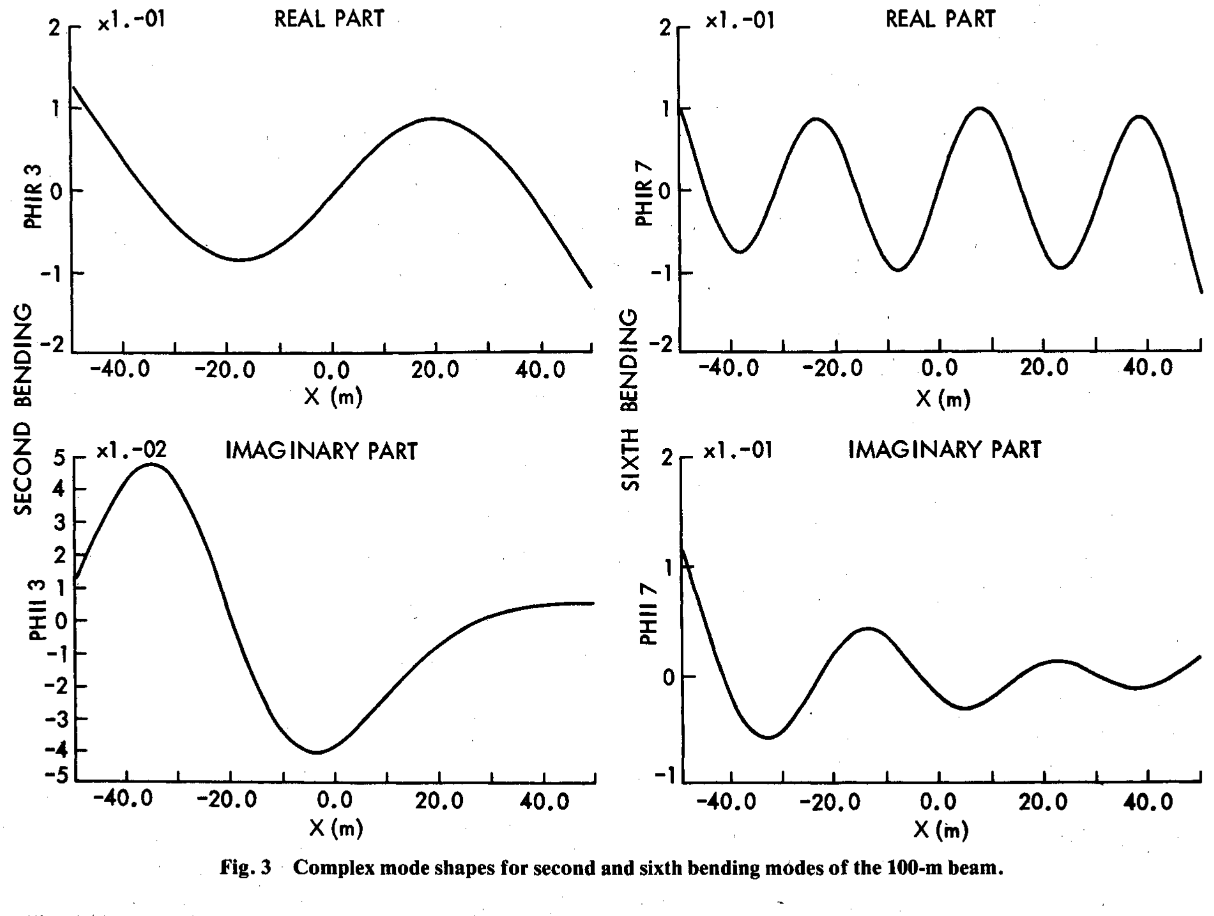

Fig. 3 Complex mode shapes for second and sixth bending modes of the 100-m beam.

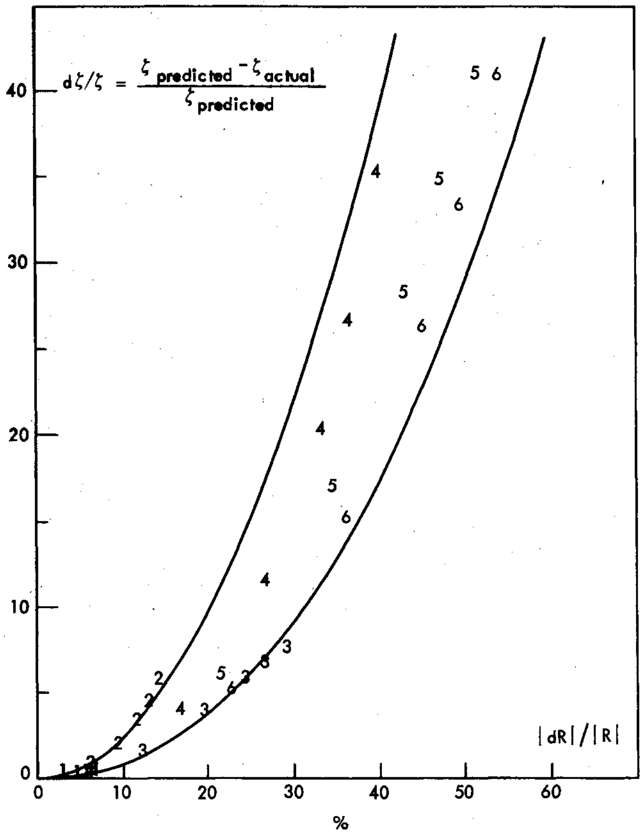

Fig. 4 Damping prediction error vs relative change in eigenvector.

As an example of this procedure, Fig. 4 shows the relative error in damping prediction as a function of the eigenvector shift for the first six modes of the 100-m beam described previously. (In this particular application, six colocated sensors/actuator pairs were used to control the beam.)

Conclusion

This paper has established analytical expressions describing the change in the modal frequencies and damping of a structure controlled by a low-authority controller. These expressions are based on a perturbation theory assuming that the relative change dR/R in the system's eigenvectors is small, but not assuming that the modes remain uncoupled. Generally speaking, the smallness of the dR/R's, which is related to the weakness of the coupling, is a measure of the validity of the perturbation formulas, and it can also be computed analytically.

The main formula shown in the paper relates the damping ratio of the nth mode to the rate feedback control gains and constitute the basis for the design of a low-authority structural control system and prediction of its performance. Although the examples presented in this paper are relatively simple for the sake of clarity, the application of the theory is of much wider scope. For instance, optimal pole placement procedures may be derived in conjunction with the basic shift formulas, as well as sensor/actuator placement optimization. The range of validity of the perturbation theory is largely dependent on the particular structure considered. From various cases studied by the author thus far, it appears that damping augmentation in a structure can be successfully designed by the present theory with damping ratios of 10% or even 20%. A general application computer program has been developed for the automatic synthesis of low-authority controllers and provides a quick way to "tune" various damping devices.

Appendix A: On the Eigenvector Perturbation Formula [(Eq. (7)]

Invertibility Theorems

Let A be a general complex square matrix of order N and λ_n its nth eigenvalue. It is assumed that λ_n has multiplicity 1, and thus there exists a unique right eigenvector R_n such that

$$AR_n = \lambda_n R_n \tag{A1}$$

Consider the operators

$$A_n \equiv A - \lambda_n I \tag{A2}$$

$$P_n \equiv R_n V_n^T \tag{A3}$$

where I is the identity and V_n any column vector such that

$$V_n^T R_n \neq 0 \tag{A4}$$

(The superscript T denotes the ordinary transposition.)

Because of Eq. (A1), A_n is singular. However, we have the following theorems.

Theorem 1: For any complex scalar $\alpha \neq 0$, $(A_n + \alpha P_n)$ has an inverse.

Proof: Choose a basis in which A is triangular. (This is always possible according to theorem 2 in Ref. 9.) The diagonal elements are the eigenvalues of A. Without loss of generality, it can always be assumed that λ_n is the first diagonal element. Since λ_n has multiplicity 1, only the first diagonal element of A_n is zero and R_n and P_n are easily found:

$$A_n = \begin{vmatrix} 0\ x \ldots\ldots & x \\ (\lambda_2 - \lambda_n) & x \\ \ddots & \vdots \\ & (\lambda_N - \lambda_n) \end{vmatrix} \quad R_n = \begin{vmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix} \quad P_n = \begin{vmatrix} V_1 & V_2 & \ldots & V_N \\ 0 & 0 & \ldots\ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots\ldots & 0 \end{vmatrix}$$

The first diagonal element of $A_n + \alpha P_n$ is therefore equal to αV_1, the others are those of A_n. Since V_1 is $\neq 0$ by Eq. (A4) and α is also $\neq 0$ by hypothesis, all the diagonal elements of $(A_n + \alpha P_n)$ (i.e., all its eigenvalues) are also $\neq 0$. Thus $(A_n + \alpha P_n)$ has an inverse.

Theorem 2: For any complex scalars α and μ such that $(\alpha + \mu\lambda_n) \neq 0$, the operator $B_n \equiv (A + \alpha P_n + \mu P_n A - \lambda_n I)$ has an inverse.

Proof: Define

$$P_n' \equiv \alpha P_n + \mu P_n A \qquad V_n' \equiv (\alpha V_n^T + \mu V_n^T A)^T$$

Then

$$B_n = (A_n + P_n') \qquad P_n' = R_n V_n'^T$$

Using Eq. (A1), one shows easily that

$$V_n'^T R_n = (\alpha + \mu\lambda_n) V_n^T R_n$$

Since this quantity is $\neq 0$ by hypothesis, it is now possible to apply theorem 1 with $\alpha = 1$ and $P_n = P_n'$.

General Eigenvector Perturbation Theorem

Theorem 3: Let P_n be the operator defined by Eqs. (A3) and (A4). The perturbation of the nth eigenvector of A is given by

$$dR_n = B_n^{-1}(P_n - I)dAR_n \tag{A5}$$

where

$$B_n = (A - P_nA + \alpha P_n - \lambda_n I) \tag{A6}$$

and α is any complex scalar $\neq \lambda_n$.

Proof: Equation (A5) is meaningful since B_n has always an inverse according to theorem 2 applied with $\mu = -1$. To prove Eq. (A5), first multiply both sides on the left by B_n. After some rearranging, one obtains

$$AdR_n + dAR_n = P_n(AdR_n + dAR_n) + \lambda_n dR_n - \alpha P_n dR_n$$

Now multiply Eq. (A6) on the left by P_n and it reduces to

$$(\alpha - \lambda_n)P_n dR_n = 0$$

Since $\alpha \neq \lambda_n$,

$$P_n dR_n = 0 \tag{A7}$$

Finally, choose

$$d\lambda_n = V_n^T(AdR_n + dAR_n) \tag{A8}$$

then it follows that

$$P_n(AdR_n + dAR_n) = R_n d\lambda_n \tag{A9}$$

and by using Eqs. (A9) and (A7) in Eq. (A6), the following relation is obtained:

$$AdR_n + dAR_n = R_n d\lambda_n + \lambda_n dR_n$$

which is nothing but the perturbation equation

$$d(A - \lambda_n R_n) = 0 \tag{A10}$$

This shows that dR_n given by Eq. (A5) satisfies the perturbation equation (A10).

Application to Eq. (7)

By choosing $\alpha = 0$ and $V_n = G_n^T R_n$, Eq. (A5) reduces to Eq. (7), and because of Eq. (A7), condition 6 is always verified. As long as one is dealing with structural modes, λ_n is $\neq 0$ and Eq. (7) is well-defined. If rigid-body modes were to be included ($\lambda_n = 0$), the more general equation (A5) should be used instead.

A particularly interesting choice for V_n is simply L_n, the left eigenvector of A. In this case,

$$B_n = (A - R_n L_n^T A + \alpha R_n L_n^T - \lambda_n I)$$

or

$$B_n = [A + (\alpha - \lambda_n) R_n L_n^T - \lambda_n I]$$

The computation of the term $P_n A_n$ which involves $2N^3$ operations is now replaced by that of $R_n L_n^T$ which requires only N^2 products.

Appendix B

Graphite-epoxy beam structural data:

Length	100 m
Outside radius	10.55 cm
Wall thickness	2.275 mm
Young's modulus	3.45 10^{11} N/m^2
Density	1607 kg/m^3

Case A (cantilevered):

Mode	Frequency	Mode shape at free end
1	6.052 − 02	1.283 − 01
2	3.793 − 01	− 1.261 − 01
3	1.062 + 00	1.242 − 01
4	2.081 + 00	− 1.221 − 01
5	3.440 + 00	1.201 − 01
6	5.139 + 00	− 1.181 − 01

Case B (free-free):

Mode	Frequency	Mode shape at tip	Center
1	0.000	2.237 − 03	2.237 − 03
2	3.851 − 01	− 6.002 − 03	− 1.030 − 17
3	1.062 + 00	− 1.015 − 02	6.975 − 03
4	2.081 + 00	− 1.420 − 02	8.344 − 15
5	3.440 + 00	− 1.826 − 02	− 1.293 − 02
6	5.139 + 00	− 2.232 − 02	− 3.153 − 12
7	7.177 + 00	− 2.638 − 02	− 1.865 − 02

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